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LETTER TO THE EDITOR

Dissipative bifurcation ratio in the area-non-preserving Hénon map†

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Abstract. The transition from periodic to chaotic behaviour via a period doubling route is studied for dissipative systems. The bifurcation ratio which characterises such a transition is calculated for the area-non-preserving Hénon map by using a simple renormalisation group method. It is seen that there exists a smooth transition from the conservative case to the dissipative case. This approximate renormalisation group calculation agrees well with the numerical result obtained recently by Zisook.

The transition from periodic to chaotic behaviour of a deterministic system has become a subject of intense research recently. The discovery of period doubling (Metropolis *et al* 1973, May 1976), universality (Feigenbaum 1978, 1979a, b), and its experimental support (Libchaber and Maurer 1980, Gollub *et al* 1981) has created even more interest.

The relevance of the ideas of the renormalisation group to the study of turbulence was first pointed out by Wilson in his study of critical phenomena. This view was reiterated by Feigenbaum (1978, 1979a, b). A theoretical basis for the utility of the renormalisation group approach was provided by Derrida *et al* (1978, 1979) when they discovered the important self-similarity property of the MSS sequences. Simple renormalisation group methods have been applied to calculate the universal bifurcation ratios in both the one-dimensional quadratic map and the two-dimensional area-preserving map (Bennetin *et al* 1980, Bountis 1981, Derrida *et al* 1979, Derrida and Pomeau 1980, Greene *et al* 1981, Grossman and Thomae 1977, Helleman 1980). The results are remarkably good.

The one-dimensional quadratic map is dissipative, whereas the two-dimensional area-preserving map is conservative. Moreover, in realistic physical systems, it is almost inevitable that some dissipation occurs. Therefore, to understand the transition between a dissipative map and a conservative one, and to provide a probably more realistic description of physical systems, it seems worthwhile to study how the bifurcation ratios change as a function of dissipation. We are therefore led to study the area-non-preserving two-dimensional Hénon map.

The general Hénon map (Hénon 1969, 1976) is described by the transformation

$$H: \begin{array}{l} x' = 1 - \mu x^2 + y \\ y' = bx. \end{array} \tag{1}$$

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The Jacobian of this transformation is equal to $-b$. When $b=-1$, it reduces to the area-preserving case. For convenience we shall call b the dissipation parameter. If we linearise the transformation in the neighbourhood of the elements x_i of an n-cycle, the map becomes

$$H^{(n)}=\prod_{i=1}^{n}\begin{pmatrix}-2\mu x_i & 1\\ b & 0\end{pmatrix}. \tag{2}$$

The eigenvalues of this matrix are

$$\lambda_n(\mu,b)=-f_n(\mu,b)\pm(f_n^2(\mu,b)-1)^{1/2}. \tag{3}$$

The lowest two cycles can be easily calculated. The elements are respectively

$$x^*=(2\mu)^{-1}\{(b-1)\pm[4\mu+(b-1)^2]^{1/2}\} \qquad (n=1), \tag{4}$$

$$\left.\begin{aligned}x_1^*&=(2\mu)^{-1}\{(1-b)\pm[4\mu-3(b-1)^2]^{1/2}\}\\ x_2^*&=(2\mu)^{-1}\{(1-b)\mp[4\mu-3(b-1)^2]^{1/2}\}\end{aligned}\right\} \qquad (n=2). \tag{5}$$

The functions f_n are therefore

$$f_1(\mu,b)=\tfrac{1}{2}\{(b-1)\pm[4\mu+(b-1)^2]^{1/2}\}, \tag{6}$$

$$f_2(\mu,b)=2\mu-2b^2+3b-2. \tag{7}$$

For systems undergoing period doubling, an approximate renormalisation group calculation consists in establishing a recursion relation by equating the eigenvalues in an n-cycle with parameter μ to those in a $2n$-cycle with parameter μ'. In this case it is equivalent to equating the f_n, i.e.

$$f_n(\mu,b)=f_{2n}(\mu',b). \tag{8}$$

The fixed point μ^* of this recursion relation then gives the approximate limit point of the period doubling sequence:

$$f_n(\mu^*,b)=f_{2n}(\mu^*,b). \tag{9}$$

The bifurcation ratio $\delta(b)$ can then be easily obtained:

$$\delta(b)=\mathrm{d}\mu(b)/\mathrm{d}\mu'(b)\,|_{\mu^*}. \tag{10}$$

The lowest-order renormalisation group calculation then gives the following quadratic equation for μ^*:

$$4\mu^{*2}-(8b^2-10b+7)\mu^*+(2b^2-2b+1)(2b^2-3b+2)=0. \tag{11}$$

The expression for $\delta(b)$ as a function of the dissipation parameter b is

$$\delta(b)=2[(b-1)^2+4\mu^*]^{1/2}. \tag{12}$$

For the special cases $b=0$ (the one-dimensional dissipative quadratic map) and $b=-1$ (the two-dimensional conservative Hénon map), the known values for δ are regained:

$$\delta(0)=5.1231 \qquad \text{(numerical 4.6692)}, \tag{13}$$

$$\delta(-1)=9.0623 \qquad \text{(numerical 8.7211)}. \tag{14}$$

For the general case, a graph of $\delta(n)$ is shown in figure 1. We see a smooth transition from the dissipative case to the conservative case as the dissipation parameter is

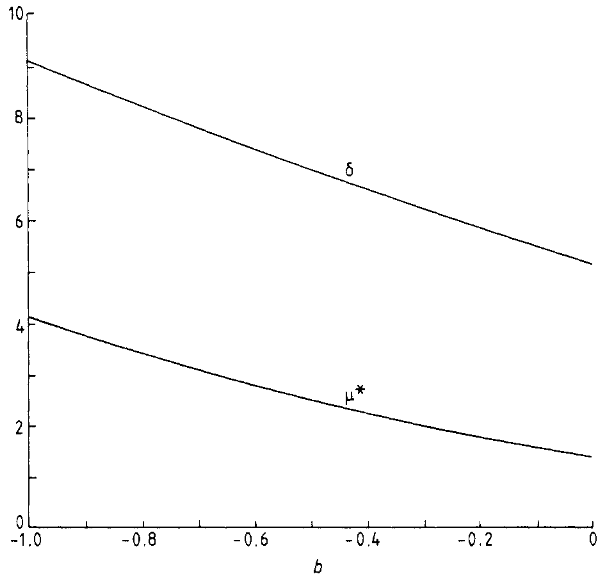

Figure 1. The bifurcation ratio δ (upper curve) and the limit point μ^* (lower curve) as a function of the dissipation parameter b.

changed. Since both dissipative and conservative systems are available for study experimentally, it will be interesting to observe this transition.

As our calculation is based on a lowest-order renormalisation group recursion relation, the result is indeed only approximate. Recently Zisook (1981) has performed a numerical calculation of the effects of dissipation. The agreement is seen to be quite good (5–10%) in view of our simple lowest-order renormalisation group calculation. From previous experience, we expect the accuracy to be substantially improved when we go to higher orders. Such a study is under way.

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